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On computing signatures of coherent systems

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It is difficult to compute the signature of a coherent system with a large number of components. This paper derives two basic formulas for computing the signature of a system which can be decomposed into two subsystems (modules). As an immediate application, we obtain the formula for computing the signature of systemwise redundancy in terms of the signatures of the original system and the backup one. The formula for computing the signature of a componentwise redundancy system is also derived. Some examples are given to illustrate the power of the main results.

1. Introduction

The notion of signature of coherent systems, introduced by Samaniego [13], is becoming more and more useful as a research tool for studying the reliability of coherent systems. Consider a coherent system consisting of n components whose lifetimes $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) with a common distribution function F. Let $T = T(X_1, \ldots, X_n)$ denote the lifetime of this system. Samaniego [13] defined the signature of the system as a probability vector $\mathbf{s} = (s_1, \ldots, s_n)$ with

$$s_i = \mathrm{P}(T = X_{i:n}), \quad i = 1, \ldots, n,$$

where $X_{i:n}$ is the ith order statistics associated to $X_1, \ldots, X_n$. Moreover, in terms of the orderings of the component lifetimes $X_1, \ldots, X_n$, he showed that

$$s_i = \frac{\text{the number of orderings for which the } i\text{th failure causes system failure}}{n!}.$$

Samaniego [13] (see also [6]) also proved that the reliability function of the system T can be expressed as a mixture of the survival functions of $X_{1:n}, \ldots, X_{n:n}$ with respect to its signature when F is continuous, that is,

$$\mathrm{P}(T > t) = \sum_{i=1}^{n} s_i \, \mathrm{P}(X_{i:n} > t). \tag{1.1}$$

The representation in (1.1) is very useful for computing the reliability of a system and for comparing coherent systems with i.i.d. components whose signatures are ordered. For more details on the notion of signature and its applications in reliability engineering, see [14,7,9,8,11,10], and the references therein.

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Although the signature is very useful for studying a coherent system with i.i.d. components, the computation of system's signature is not simple, especially when the number of components is large. Generally, the signature of a system can be obtained from the well-known cut set representation of the system lifetime. An alternate approach provided by Boland [2] is based on determining the number of path sets in the system and the number of what he called ordered cut sets, which will be discussed in detail in Section 3. Besides, Triantafyllou and Koutras [15] proposed a method based on a generating function approach.

In practical reliability analysis, a common procedure is to compute first the reliability of each of the disjoint subsystems comprising a system, and then compute the overall system reliability from these subsystems' reliabilities (see [1], p. 16). Motivated by this, we consider computing the signatures of some complex systems by its subsystem signatures. We derive two basic formulas for computing the signature of a system which can be decomposed into two subsystems in the sense of parallel and series, respectively. Using these two formulas, the computation of signatures of some complex systems becomes a lot simpler. As an immediate application, we obtain the formula for computing the signature of systemwise redundancy in terms of the signatures of the original system and the backup one. Besides, a formula for the signature of system with redundancy at component level is derived as well. Throughout, it is assumed that all component lifetimes of any system are i.i.d. with a common continuous distribution function F.

The paper is organized as follows. In Section 2, we derive two basic formulas for computing the signature of a system which can be decomposed into two subsystems (modules). In Section 3, formulas for computing the signatures of systemwise and componentwise redundancy systems are derived. In Section 4, some examples are given to illustrate the power of the main results in Sections 2 and 3. Finally, in Section 5, we make a summary on this research.

2. Two basic formulas

In this section, we derive two basic formulas for computing the signature of a coherent system based on given signatures of its subsystems (modules). For the rigorous definitions of *module* and *modular decomposition* of coherent systems, one may refer to Definitions 4.1 and 4.3 in [1], Chap.1.

Consider a coherent system (C, ϕ), where $C = \{1, \ldots, n+m\}$ is the index set of components, and ϕ is its structure function. Let X_i denote the lifetime of the component i for $i = 1, \ldots, n+m$, and $X_1, \ldots, X_{n+m}$ be i.i.d. with a continuous distribution function F. Suppose that $\{(A, \chi_1), (B, \chi_2)\}$ is a modular decomposition of the system (C, ϕ), where $A = \{1, \ldots, n\}, B = \{n+1, \ldots, n+m\}$, and χ_1 and χ_2 are the corresponding structure functions. Also, suppose that the overall system is either the parallel or the series of the modules (A, χ_1) and (B, χ_2). Let $\mathbf{p} = (p_1, \ldots, p_n)$ and $\mathbf{q} = (q_1, \ldots, q_m)$ be the signature vectors of the modules (A, χ_1) and (B, χ_2), respectively. Now, let us determine the signature vector $\mathbf{s}$ of the overall system based on signatures $\mathbf{p}$ and $\mathbf{q}$.

We first consider a coherent system which is the parallel of two modules.

Theorem 2.1. *Suppose that the coherent system (C, ϕ) is the parallel of the modules (A, χ_1) and (B, χ_2) with $n \le m$. Then the overall system has a signature vector $\mathbf{s}$ with ith component s_i given by*

$$s_1 = 0;$$

for $2 \le i \le n$,

$$s_i = \frac{\sum_{j=1}^{i-1} \binom{i-1}{j} \left[\left(p_{i-j} \sum_{k=1}^{j} q_k \right) \binom{n+m-i}{m-j} + \left(q_{i-j} \sum_{k=1}^{j} p_k \right) \binom{n+m-i}{n-j} \right]}{\binom{n+m}{n}}; \tag{2.1}$$

for $n < i \le m$,

$$s_i = \frac{\sum_{j=i-n}^{i-1} \left(p_{i-j} \sum_{k=1}^{j} q_k \right) \binom{i-1}{j} \binom{n+m-i}{m-j} + \sum_{j=1}^{n} \left(q_{i-j} \sum_{k=1}^{j} p_k \right) \binom{i-1}{j} \binom{n+m-i}{n-j}}{\binom{n+m}{n}}; \tag{2.2}$$

and for $m < i \le n+m$,

$$s_i = \frac{\sum_{j=i-n}^{m} \left(p_{i-j} \sum_{k=1}^{j} q_k \right) \binom{i-1}{j} \binom{n+m-i}{m-j} + \sum_{j=i-m}^{n} \left(q_{i-j} \sum_{k=1}^{j} p_k \right) \binom{i-1}{j} \binom{n+m-i}{n-j}}{\binom{n+m}{n}}. \tag{2.3}$$

Proof. First, denote by $T = \phi(X_1, \ldots, X_{n+m})$, $T_A = \chi_1(X_1, \ldots, X_n)$ and $T_B = \chi_2(X_{n+1}, \ldots, X_{n+m})$ the lifetimes of the system (C, ϕ) and the subsystems (A, χ_1) and (B, χ_2), respectively. Suppose that the underlying probability space is $(\Omega, \mathscr{A}, \mathsf{P})$, and denote by $\mathcal{P}_{n+m}$ the permutation group of $(1, 2, \ldots, n+m)$. For each $i \in \{1, \ldots, n+m\}$, define the subset E_i of $\mathcal{P}_{n+m}$ as follows:

$$E_i = \left\{ \boldsymbol{\pi} = (\pi_1, \ldots, \pi_{n+m}) \in \mathcal{P}_{n+m} : \begin{array}{l} \phi(\mathbf{x}) = x_{\pi_i} \text{ whenever } x_{\pi_1} < x_{\pi_2} < \cdots < x_{\pi_{n+m}} \\ \text{for all } \mathbf{x} = (x_1, \ldots, x_{n+m}) \in \Re_+^{n+m} \end{array} \right\}.$$

Since (C, ϕ) is a coherent system, for given $\boldsymbol{\pi}$, $\phi(\mathbf{x})$ depends on the magnitude of the x_i's only through the order $\boldsymbol{\pi}$. So, the set E_i is well defined. On the other hand, E_i can be divided into two subsets E_i^1 and E_i^2 as below

$$E_i^1 = \{\boldsymbol{\pi} \in E_i : \pi_i \in A\}, \qquad E_i^2 = \{\boldsymbol{\pi} \in E_i : \pi_i \in B\}.$$

Next, for each $\boldsymbol{\pi} \in \mathcal{P}_{n+m}$, define

$$H_{\boldsymbol{\pi}} = \left\{ w \in \Omega : X_{\pi_1}(w) < X_{\pi_2}(w) < \cdots < X_{\pi_{n+m}}(w) \right\},$$

and

$$H_0 = \Omega \setminus \left(\bigcup_{\boldsymbol{\pi} \in \mathcal{P}_{n+m}} H_{\boldsymbol{\pi}} \right).$$

Since the X_i's are assumed to be i.i.d. with a continuous distribution function, $\mathsf{P}(H_0) = 0$. Clearly, $H_{\boldsymbol{\pi}} \cap H_{\boldsymbol{\sigma}} = \emptyset$ for each pair $\boldsymbol{\pi} \neq \boldsymbol{\sigma}$, and

$$\mathsf{P}(H_{\boldsymbol{\pi}}) = \frac{1}{(n+m)!}, \quad \forall \boldsymbol{\pi} \in \mathcal{P}_{n+m}.$$

Also, for each $i \in \{1, \ldots, n+m\}$,

$$\{T = X_{i:n+m}\} = \bigcup_{\boldsymbol{\pi} \in E_i} H_{\boldsymbol{\pi}},$$

and, hence,

$$s_i = \mathsf{P}(T = X_{i:n+m}) = \frac{|E_i|}{(n+m)!}, \tag{2.4}$$

where $|E_i|$ denotes the cardinality of set E_i.

Since the system ϕ is the parallel of two subsystems, it is obvious that $s_1 = 0$. To derive the formulas for other s_i's, we need to compute the cardinality $|E_i| = |E_i^1| + |E_i^2|$. For fixed $i \geq 2$ and each $\boldsymbol{\pi} \in E_i^1$, denote by j the number of π_ℓ, $1 \leq \ell < i$, such that $\pi_\ell \in B$, that is,

$$j = |\{\pi_\ell : \pi_\ell \in B, 1 \leq \ell < i\}|. \tag{2.5}$$

Clearly, $1 \leq j < i$. Now, consider the following three cases.

Case 1. $2 \leq i \leq n$: First let us compute the cardinality $|E_i^1|$. For $1 \leq j < i$, define

$$E_i^{1j} = \left\{ \boldsymbol{\pi} \in E_i^1 : \{\pi_1, \ldots, \pi_{i-1}\} \text{ contains } j \text{ elements of } B \right\}.$$

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ and $\widetilde{X}_{1:m} \leq \widetilde{X}_{2:m} \leq \cdots \leq \widetilde{X}_{m:m}$ be the order statistics corresponding to two sets of random variables $\{X_1, \ldots, X_n\}$ and $\{X_{n+1}, \ldots, X_{n+m}\}$. For $\boldsymbol{\pi} \in E_i^{1j}$, $w \in H_{\boldsymbol{\pi}}$ if and only if

$$T_B(w) \leq \widetilde{X}_{j:m}(w), \qquad T_A(w) = X_{i-j:n}(w) \quad \text{and} \quad \widetilde{X}_{j:m}(w) < X_{i-j:n}(w).$$

Thus,

$$\bigcup_{\boldsymbol{\pi} \in E_i^{1j}} H_{\boldsymbol{\pi}} = \left\{ T_B \leq \widetilde{X}_{j:m}, T_A = X_{i-j:n}, \widetilde{X}_{j:m} < X_{i-j:n} < \widetilde{X}_{j+1:m} \right\}. \tag{2.6}$$

Since

$$E_i^1 = \bigcup_{j=1}^{i-1} E_i^{1j} \quad \text{and} \quad E_i^{1,j_1} \bigcap E_i^{1,j_2} = \emptyset \quad \text{for } 1 \leq j_1 \neq j_2 < i, \tag{2.7}$$

we have

$$\mathsf{P}\left(\bigcup_{\boldsymbol{\pi} \in E_i^{1j}} H_{\boldsymbol{\pi}} \right) = \frac{|E_i^{1j}|}{(n+m)!}. \tag{2.8}$$

On the other hand,

$$
\begin{aligned}
&\mathrm{P}\left(T_B \le \widetilde{X}_{j:m}, T_A = X_{i-j:n},\ \widetilde{X}_{j:m} < X_{i-j:n} < \widetilde{X}_{j+1:m}\right)\\
&\quad = \mathrm{P}\left(T_B \le \widetilde{X}_{j:m}, T_A = X_{i-j:n} | \widetilde{X}_{j:m} < X_{i-j:n} < \widetilde{X}_{j+1:m}\right) \mathrm{P}\left(\widetilde{X}_{j:m} < X_{i-j:n} < \widetilde{X}_{j+1:m}\right)\\
&\quad = \mathrm{P}\left(T_B \le \widetilde{X}_{j:m}, T_A = X_{i-j:n}\right) \cdot \mathrm{P}\left(\widetilde{X}_{j:m} < X_{i-j:n} < \widetilde{X}_{j+1:m}\right)\\
&\quad = \mathrm{P}(T_B \le \widetilde{X}_{j:m}) \cdot \mathrm{P}(T_A = X_{i-j:n}) \cdot \mathrm{P}\left(\widetilde{X}_{j:m} < X_{i-j:n} < \widetilde{X}_{j+1:m}\right)\\
&\quad = \left(\sum_{k=1}^{j} q_k\right) \cdot p_{i-j} \cdot \frac{\binom{i-1}{j}\binom{n+m-i}{m-j} m!n!}{(n+m)!},
\end{aligned} \tag{2.9}
$$

where the last equality follows from the definition of the signatures p_{i-j} and q_k, and Eq. (2.9) follows from the independence between $\{T_B \le \widetilde{X}_{j:m}, T_A = X_{i-j:n}\}$ and $\{\widetilde{X}_{j:m}, \widetilde{X}_{j+1:m}, X_{i-j:n}\}$ since the former depends on the X_k's only through the ranks of the two samples $\{X_1, \dots, X_n\}$ and $\{X_{n+1}, \dots, X_{n+m}\}$; see Lemma 8.3.11 of Randles and Wolfe [12]. From (2.6)–(2.9), it follows that

$$
|E_i^1| = \sum_{j=1}^{i-1} |E_i^{1j}| = \sum_{j=1}^{i-1} \binom{i-1}{j}\binom{n+m-i}{m-j}\left(m!\sum_{k=1}^{j} q_k\right)(n!p_{i-j}). \tag{2.10}
$$

Similarly,

$$
\left|E_i^2\right| = \sum_{j=1}^{i-1} \binom{i-1}{j}\binom{n+m-i}{n-j}\left(n!\sum_{k=1}^{j} p_k\right)(m!q_{i-j}). \tag{2.11}
$$

The desired result (2.1) now follows from (2.4), (2.10) and (2.11).

Case 2. $n < i \le m$: The proof is similar to Case 1. In this case, from the definition of j in (2.5), it can be seen that $j \in \{i-n, i-n+1, \dots, i-1\}$. Thus, the sum $\sum_{j=1}^{i-1}$ in (2.10) is replaced by $\sum_{j=i-n}^{i-1}$. Similarly, the sum $\sum_{j=1}^{i-1}$ in (2.11) is replaced by $\sum_{j=1}^{n}$.

Case 3. $m < i \le n+m$: Similarly, from (2.5), the sums $\sum_{j=1}^{i-1}$ in (2.10) and (2.11) are replaced by $\sum_{j=i-n}^{m}$ and $\sum_{j=i-m}^{n}$, respectively.

Combining the above three cases, we complete the proof of the theorem. □

The next result illustrates how to compute the signature of a coherent system which is the series of two modules.

Theorem 2.2. *Suppose that the overall system (C, ϕ) is the series of the modules (A, χ_1) and (B, χ_2) with $n \le m$. Then the signature vector* **s** *of the overall system is given by*

$$s_{n+m} = 0;$$

for $1 \le i \le n$,

$$
s_i = \frac{\sum_{j=0}^{i-1}\binom{i-1}{j}\left[\left(p_{i-j}\sum_{k=j+1}^{m} q_k\right)\binom{n+m-i}{m-j} + \left(q_{i-j}\sum_{k=j+1}^{n} p_k\right)\binom{n+m-i}{n-j}\right]}{\binom{n+m}{n}}; \tag{2.12}
$$

for $n < i \le m$,

$$
s_i = \frac{\sum_{j=i-n}^{i-1}\left(p_{i-j}\sum_{k=j+1}^{m} q_k\right)\binom{i-1}{j}\binom{n+m-i}{m-j} + \sum_{j=0}^{n-1}\left(q_{i-j}\sum_{k=j+1}^{n} p_k\right)\binom{i-1}{j}\binom{n+m-i}{n-j}}{\binom{n+m}{n}}; \tag{2.13}
$$

and for $m < i < n+m$,

$$
s_i = \frac{\sum_{j=i-n}^{m-1}\left(p_{i-j}\sum_{k=j+1}^{m} q_k\right)\binom{i-1}{j}\binom{n+m-i}{m-j} + \sum_{j=i-m}^{n-1}\left(q_{i-j}\sum_{k=j+1}^{n} p_k\right)\binom{i-1}{j}\binom{n+m-i}{n-j}}{\binom{n+m}{n}}. \tag{2.14}
$$

Proof. The proof is very similar to that of Theorem 2.1. Since $s_{n+m} = 0$ is obvious, we only need to compute the signatures s_i for $i = 1, \ldots, n+m-1$. It suffices to consider the following three cases:

Case 1. $1 \le i \le n$;

Case 2. $n < i \le m$;

Case 3. $m < i < n+m$.

For Case 1, unlike the parallel structure in Theorem 2.1, $\left(m! \sum_{k=1}^{j} q_k\right)$ in (2.10) should be replaced by $\left(m! \sum_{k=j+1}^{m} q_k\right)$, and j should be summed from 0 to $i-1$ because as the component whose failure causes the overall system to fail at ith failure time of components is from the module A, the module B must be working at that time. The rest of the arguments are the same as in Case 1 of Theorem 2.1. For Cases 2 and 3, pay more attention to the appropriate range for j defined in (2.5). We omit the details. □

3. On redundancy systems

As a very useful method for increasing the reliability of a coherent system, redundancy has been widely applied in reliability engineering. Often there are two ways, systemwise redundancy and componentwise redundancy. Suppose there is a coherent system of order n and one has the opportunity to enhance its performance by incorporating redundancy of n identical spares for the components. Systemwise redundancy is to place the n redundant components as an identical system in parallel with the original system, and componentwise redundancy is to place every redundant component separately in parallel with every component in the original system. A well-known principle is that componentwise redundancy is more effective than systemwise redundancy (see [1,3]).

In this section, we discuss how to determine the signature of a coherent system with systemwise redundancy and with componentwise redundancy from the signature of the original system, respectively.

As an immediate consequence of Theorem 2.1, we obtain the formula for computing the signature of a coherent system with systemwise redundancy.

Proposition 3.1. *If $\mathbf{p} = (p_1, \ldots, p_n)$ is the signature vector of a coherent system, then the system with systemwise redundancy has a signature vector $\mathbf{s} = (s_1, \ldots, s_{2n})$ with ith component given by*

$$s_1 = 0,$$

$$s_i = \frac{\sum_{j=1}^{i-1} \binom{i-1}{j} \binom{2n-i}{n-j} \left(2p_{i-j} \sum_{k=1}^{j} p_k\right)}{\binom{2n}{n}} \quad \textit{for } 1 \le i \le n \tag{3.1}$$

and

$$s_i = \frac{\sum_{j=i-n}^{n} \binom{i-1}{j} \binom{2n-i}{n-j} \left(2p_{i-j} \sum_{k=1}^{j} p_k\right)}{\binom{2n}{n}} \quad \textit{for } n < i \le 2n. \tag{3.2}$$

A k-out-of-n system functions if and only if at least k of its n components function. A series system is an n-out-of-n system, and a parallel system is a 1-out-of-n system. Kochar et al. [6] established a formula for the evaluation of the signature of a k-out-of-n system with systemwise redundancy. Of course, that can be obtained from Proposition 3.1 by straightforward discussions and simplifications.

Corollary 3.2. *The signature vector $\mathbf{s} = (s_1, \ldots, s_{2n})$ of a k-out-of-n system with systemwise redundancy is given as follows:*

$$s_{2n-2k+2+r} = \frac{2\binom{2n-2k+1+r}{n-k+1+r}\binom{2k-2-r}{k-1-r}}{\binom{2n}{n}} \quad \textit{for } r = 0, 1, \ldots, k-1, \tag{3.3}$$

and $s_i = 0$ for $1 \le i < 2n-2k+2$ or $2n-k+1 < i \le 2n$.

Proof. The signature vector of a k-out-of-n system is

$$\mathbf{p} = (0, \ldots, 0, 1_{n-k+1}, 0, \ldots, 0)$$

with 1 being the $(n-k+1)$th element of the vector. Observing (3.1) and (3.2), we only concentrate our attention on $j^* = i - (n-k+1)$ in the summation of the numerators in (3.1) and (3.2) for a given i since $p_{i-j} = 0$ otherwise. Clearly,

(i) if $1 < i < 2n-2k+2$, then $j^* \le n-k$ and hence $\sum_{k=1}^{j^*} p_k = 0$, so $s_i = 0$;
(ii) if $2n-k+1 < i \le 2n$, then $j^* \ge n+1$, so $s_i = 0$;
(iii) for $r = 0, 1, \ldots, k-1$, if $i = 2n-2k+2+r \le n$, we have

$$j^* = n-k+1+r < 2n-2k+1+r = i-1$$

and $j^* \ge n-k+1$, and then $\sum_{k=1}^{j^*} p_k = 1$, hence (3.3) holds; if $i = 2n-2k+2+r > n$, we have

$$j^* = n-k+1+r > n-2k+2+r = i-n$$

and $j^* \ge n-k+1$, then $\sum_{k=1}^{j^*} p_k = 1$ and hence (3.3) holds.

This completes the proof of the corollary. □

Although (3.3) is different from (27) of [6], one can easily verify that they are equivalent.

Next, we compute the signature of a coherent system with componentwise redundancy. As mentioned in Section 1, Boland [2] provided an approach to computing the signatures of coherent systems based on determining the number of path sets of the system. The following lemma presents the aforementioned technique, which plays a crucial role in the proof of Theorem 3.4.

Lemma 3.3 (*[2]*). *Let r_i be the number of path sets of size i of a coherent system with n i.i.d. components and let $\mathbf{s} = (s_1, \ldots, s_n)$ be the signature vector of this system. Then*

$$r_i = \binom{n}{i}\left(\sum_{j=n-i+1}^{n} s_j\right) \quad \text{for } i = 1, \ldots, n,$$

and

$$s_{n-j} = \frac{r_{j+1}}{\binom{n}{j+1}} - \frac{r_j}{\binom{n}{j}} \quad \text{for } j = 0, \ldots, n-1.$$

Theorem 3.4. *Let $\mathbf{p} = (p_1, \ldots, p_n)$ be the signature vector of a coherent system, and define $d_0 = 0$ and*

$$d_i = \sum_{j=\lfloor \frac{i-1}{2} \rfloor + 1}^{i \wedge n} 2^{2j-i} \binom{j}{i-j} \binom{n}{j} \left(\sum_{l=n-j+1}^{n} p_l\right) \quad \text{for } i = 1, \ldots, 2n, \tag{3.4}$$

where $i \wedge n = \min\{i, n\}$, and $\lfloor x \rfloor$ is the largest integer not greater than x. Then the system with componentwise redundancy has a signature vector $\mathbf{s} = (s_1, \ldots, s_{2n})$ given by

$$s_{2n-i} = \frac{d_{i+1}}{\binom{2n}{i+1}} - \frac{d_i}{\binom{2n}{i}} \quad \text{for } i = 0, \ldots, 2n-1. \tag{3.5}$$

Proof. According to Lemma 3.3, we just need to determine the number of path sets of size i of the redundancy system for $i = 1, \ldots, 2n$. Here we regard the redundancy system as being composed of n modules, each being a parallel system containing two components — the original component and its identical back up one. Let Q_i denote the collection of all path sets of size i of the redundancy system for $i = 1, \ldots, 2n$. Observe that, for any path set (not necessary a minimal path set) in Q_i, there exists some unique $j \in \{\lfloor \frac{i-1}{2} \rfloor + 1, \ldots, i \wedge n\}$, such that components of the path set are from j different modules, and all original components of the j modules must constitute a path set of size j of the original system. For each $1 \le i \le 2n$, set $j_0(i) = \lfloor \frac{i-1}{2} \rfloor$, and denote by Q_i^j the collection of all path sets in Q_i, whose components are from j different modules, where $j \in \{j_0(i) + 1, \ldots, i \wedge n\}$. Thus,

$$Q_i = \bigcup_{j=j_0(i)+1}^{i \wedge n} Q_i^j,$$

and $Q_i^j, j = j_0(i), \ldots, i \wedge n$, are pairwise mutually exclusive. Hence, the number of path sets of size i of the redundancy system is given by

$$d_i = \sum_{j=j_0(i)+1}^{i \wedge n} |Q_i^j|, \quad i = 1, \ldots, 2n.$$

Now, the problem reduces to finding $|Q_i^j|$, which is the same as the number of all possible outcomes in choosing i components from j different modules, while ensuring that each of the modules has at least one component to be chosen. If

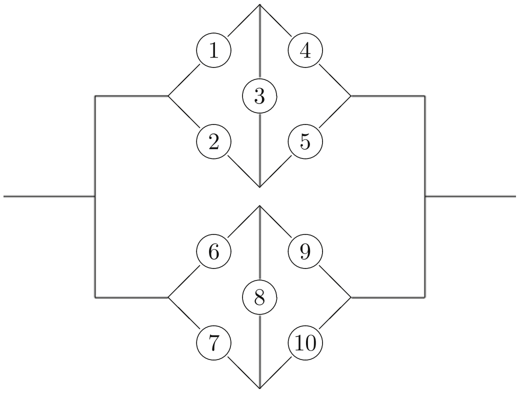

Fig. 1. The parallel of two bridge structures with signature vector $\mathbf{s}_1 = (0, 0, 0, 2/105, 34/315, 86/315, 3/10, 19/90, 4/45, 0)$.

we denote by b_k the number of all path sets of size k of the original system, $k = 1, \ldots, n$, it is not hard to obtain that, for $i \in \{1, \ldots, 2n\}$ and $j \in \{j_0(i) + 1, \ldots, i \wedge n\}$,

$$|Q_i^j| = 2^{2j-i} \binom{j}{i-j} b_j.$$

By Lemma 3.3, we have

$$b_j = \binom{n}{j} \left(\sum_{l=n-j+1}^{n} p_l \right),$$

and hence (3.4) and (3.5) are easily obtained. This completes the proof. □

Kochar et al. [6] derived the signature vector $\mathbf{s} = (s_1, \ldots, s_{2n})$ of a k-out-of-n system with redundancy at the component level:

$$s_{2n-2k+2+r} = \frac{\binom{n-1}{k-1}\binom{k-1}{r} 2^r}{\binom{2n-1}{2k-2-r}} \quad \text{for } r = 0, 1, \ldots, k-1,$$

with $s_i = 0$ for $1 \le i < 2n - 2k + 2$ and for $2n - k + 1 < i \le 2n$. From Theorem 3.4, we can obtain another formula to calculate the signature of a k-out-of-n system with componentwise redundancy. It is not easy to shown that these two formulas are equivalent. It is worth mentioning the following equality:

$$\sum_{j=\lfloor \frac{i-1}{2} \rfloor + 1}^{i \wedge n} 2^{2j-i} \binom{j}{i-j} \binom{n}{j} = \binom{2n}{i}, \quad i = 1, \ldots, 2n,$$

which can be seen from the last paragraph in the proof of Theorem 3.4.

4. Examples

In this section, three coherent systems are given as illustrations of the main results in Sections 2 and 3. It can be easily observed that our formulas are very useful for computing the signature of a coherent system with a large number of components.

Example 4.1. The coherent systems in Figs. 1 and 2 are the parallel and the series of two bridge structures. It is known that the signature of the bridge structure is $\mathbf{p} = (0, 1/5, 3/5, 1/5, 0)$. By Proposition 3.1 and Theorem 2.2, the signatures of the parallel and the series of two bridge structures are, respectively, given by

$$\mathbf{s}_1 = \left(0, 0, 0, \frac{2}{105}, \frac{34}{315}, \frac{86}{315}, \frac{3}{10}, \frac{19}{90}, \frac{4}{45}, 0\right) \tag{4.1}$$

and

$$\mathbf{s}_2 = \left(0, \frac{4}{45}, \frac{19}{90}, \frac{3}{10}, \frac{86}{315}, \frac{34}{315}, \frac{2}{105}, 0, 0, 0\right). \tag{4.2}$$

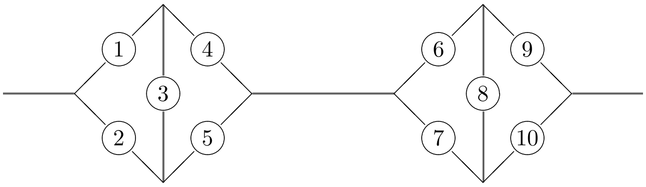

Fig. 2. The series of two bridge structures with signature vector $\mathbf{s}_2 = (0, 4/45, 19/90, 3/10, 86/315, 34/315, 2/105, 0, 0, 0)$.

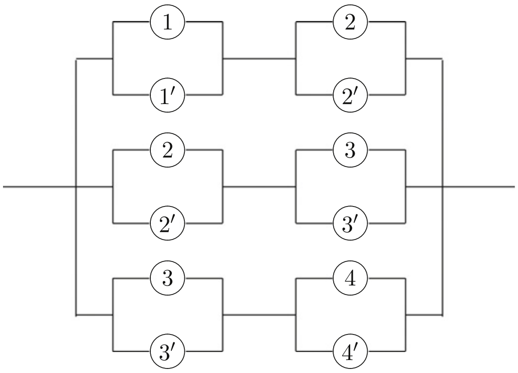

Fig. 3. Linear consecutive 2-out-of-4: *G* system with componentwise redundancy.

From (4.1) and (4.2), it seems that the parallel and the series systems of two bridge structures are dual. In fact, this is true, as shown below. For any $\mathbf{x} \in \{0, 1\}^{10}$, the structure function of the parallel of two bridges indicated in Fig. 1 is given by

$$\phi(\mathbf{x}) = [(x_1 \amalg x_2)(x_1 \amalg x_3 \amalg x_5)(x_2 \amalg x_3 \amalg x_4)(x_4 \amalg x_5)] \amalg \ [(x_6 \amalg x_7)(x_6 \amalg x_8 \amalg x_{10})(x_7 \amalg x_8 \amalg x_9)(x_9 \amalg x_{10})],$$

where $x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2)$. So, we have

$$\phi(1 - \mathbf{x}) = [(1 - x_1x_2)(1 - x_1x_3x_5)(1 - x_2x_3x_4)(1 - x_4x_5)] \amalg \ [(1 - x_6x_7)(1 - x_6x_8x_{10})(1 - x_7x_8x_9)(1 - x_9x_{10})]$$

and, then,

$$\begin{aligned} 1 - \phi(1 - \mathbf{x}) &= [1 - (1 - x_1x_2)(1 - x_1x_3x_5)(1 - x_2x_3x_4)(1 - x_4x_5)] \\ &\quad \times [1 - (1 - x_6x_7)(1 - x_6x_8x_{10})(1 - x_7x_8x_9)(1 - x_9x_{10})] \\ &= [(x_1x_2) \amalg (x_1x_3x_5) \amalg (x_2x_3x_4) \amalg (x_4x_5)][(x_6x_7) \amalg (x_6x_8x_{10}) \amalg (x_7x_8x_9) \amalg (x_7x_{10})]. \end{aligned}$$

Note that $1 - \phi(1 - \mathbf{x})$ is just the structure function of the series of two bridges indicated in Fig. 2 with an exchange of positions of components 2 and 4 as well as components 7 and 9. This completes the proof.

Example 4.2. A linear consecutive k-out-of-n: G system is a system of n linearly ordered components which functions if and only if at least k consecutive components function. For reviews of this kind of systems, see [5,4]. Let the modules A and B denote a linear consecutive 2-out-of-4: G system and a linear consecutive 3-out-of-5: G system, respectively. It is known from [7] that for these two systems the signature vectors are given by

$$\mathbf{p} = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right) \quad \text{and} \quad \mathbf{q} = \left(\frac{1}{5}, \frac{1}{2}, \frac{3}{10}, 0, 0\right).$$

Therefore, by applying Theorem 2.1, the signature vector of the overall system is

$$\mathbf{s}_3 = \left(0, 0, \frac{1}{28}, \frac{41}{252}, \frac{17}{63}, \frac{17}{63}, \frac{5}{28}, \frac{21}{252}, 0\right).$$

Example 4.3. A linear consecutive 2-out-of-4: G system with componentwise redundancy is depicted in Fig. 3. It is known from Example 4.2 that the signature of the original system is $\mathbf{p} = (0, 1/2, 1/2, 0)$. By Theorem 3.4, the signature vector of the redundancy system is given by

$$\mathbf{s}_4 = \left(0, 0, 0, \frac{3}{70}, \frac{6}{35}, \frac{5}{14}, \frac{3}{7}, 0\right).$$

5. Conclusions

In practical reliability analysis, a large number of systems with numerous components are the composition of some disjoint subsystems. In this paper, we establish two basic formulas for computing the signatures of such systems based on the signatures of the subsystems. As a very useful method for increasing the reliability of a coherent system, redundancy is very common in reliability engineering. We also compute the signatures of systems with redundancy at the system and at the component levels. All the formulas derived in this paper can greatly reduce the workload of computing signatures of coherent systems with numerous components. How to determine the signature of a coherent system from the signatures of its non-disjoint subsystems is left for future research.

Navarro and Eryilmaz [7] introduced the minimal and the maximal signatures of a coherent system with exchangeable components, which allow us to represent system distribution as generalized mixtures (i.e., mixtures with possibly negative weights) of series and parallel systems. It is still an open problem how to calculate the minimal and the maximal signatures of a coherent system based on corresponding given signatures of its subsystems.

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